Equivalence in spin wave theories

1994 J. Phys. A: Math. Gen. 27 2915

(http://iopscience.iop.org/0305-4470/27/8/027)

J. Phys. A: Math. Gen. 27 (1994) 2915–2922. Printed in the UK

Equivalence in spin wave theories

Chigak Itoi† and Masahide Kato‡

Department of Physics and Atomic Energy Research Institute, College of Science and Technology, Nihon University, Kanda Surugadai, Chiyoda-ku, Tokyo 101, Japan

Received 15 November 1993

Abstract. Equivalence in spin wave theories is pointed out from the viewpoint of gauge invariance. Both the Holstein–Primakoff and the Dyson–Maleev representations are derived by imposing certain gauge fixing conditions on the Schwinger formalism which is gauge invariant. We show explicitly the equality of partition functions of the Heisenberg model calculated in the Holstein–Primakoff and the Dyson–Maleev representations in high-temperature expansion.

1. Introduction

Spin wave theory is well known as a traditional method to study critical phenomena of quantum spin systems. Spin operators are represented in terms of bosonic creation and annihilation operators. In this theory, thermodynamic quantities can be calculated in a large spin expansion. There are several types of representations in spin wave theory, say Holstein–Primakoff [1], Dyson–Maleev [2, 3, 5] and Schwinger [4, 8] representations. In the Holstein–Primakoff representation, spin operators are

$$S^+ = \sqrt{2s - b^\dagger b}\, b \qquad S^- = b^\dagger \sqrt{2s - b^\dagger b} \qquad S^z = s - b^\dagger b \tag{1.1}$$

where boson operators b and $b^\dagger$ satisfy $[b, b^\dagger] = 1$, and s is a magnitude of spin. In this representation, the Hamiltonian takes a non-polynomial form because of the square root functions $S^\pm$. In the Dyson–Maleev representation, spin operators are

$$S^+ = (2s - b^\dagger b)b \qquad S^- = b^\dagger \qquad S^z = s - b^\dagger b\,. \tag{1.2}$$

One might be suspicious of unitarity of this representation, since $(S^-)^\dagger \neq S^+$, and then the Hamiltonian does not look Hermitian. This problem was discussed by Oguchi and Honma [6, 7]. They found a non-unitary transformation between the Holstein–Primakoff and the Dyson–Maleev representations. This implies that the Dyson–Maleev representation may not be equivalent to the Holstein–Primakoff one.

In the Schwinger representation, two kinds of bosons are introduced. Spin operators are

$$S^+ = a^\dagger b \qquad S^- = b^\dagger a \qquad S^z = \tfrac{1}{2}(a^\dagger a - b^\dagger b) \tag{1.3}$$

where a and b denote spin up and down annihilation operators, respectively. A constraint with respect to the number operator

$$a^\dagger a + b^\dagger b - 2s = 0$$

† E-mail address: itoi@phys.cst.nihon-u.ac.jp

‡ E-mail address: kato@phys.cst.nihon-u.ac.jp

0305-4470/94/082915+08$19.50

restricts the states into a spin s representation. These three representations actually look different from each other, even if every representation satisfies the same spin commutation relation. Although there have been many works on critical phenomena of quantum spin systems in these spin wave theories, it is still unclear whether or not obtained results depend on the choice of representation. The relation between different representations should be clarified in order to obtain universal results.

In this paper, equivalence in spin wave theories is clarified. From the viewpoint of gauge invariance, it becomes obvious that the Holstein–Primakoff, the Dyson–Maleev and the Schwinger representations are equivalent to each other. We point out that the Schwinger representation is invariant under a local $U(1)$ gauge transformation which is generated by the constraint function $a^\dagger a+b^\dagger b-2s$ at each lattice site. In the well known Dirac method [9] for a gauge invariant theory, a gauge fixing condition is introduced to eliminate redundant gauge degrees of freedom. All gauge invariant quantities calculated in one gauge fixing condition agree with those in other gauge fixing condition, because of the gauge invariance of the original theory. We can choose any gauge fixing condition if it does not commute with the constraint function. We show that both the Holstein–Primakoff and the Dyson–Maleev representations are obtained from the Schwinger representation with different gauge fixing conditions. Thus, these representations are unitary equivalent to each other with respect to gauge invariant quantities, even though a non-unitary transformation connects gauge dependent quantities in the Dyson–Maleev representation to those in others. We confirm this argument in practical calculation of the partition function which is a gauge invariant quantity.

This paper is organized as follows. In section 2, we treat the Heisenberg model in the Schwinger representation, which possesses a local $U(1)$ gauge invariance. Certain gauge fixing conditions give us the Holstein–Primakoff and the Dyson–Maleev representations. We define a new inner product for unitarity of the Dyson–Maleev representation. In section 3, the identity of partition functions in the two representations is shown explicitly in high temperature expansion.

2. Gauge invariance and gauge fixing

First, we discuss gauge invariance of a system which consists of only one spin operator. The idea will be extended to a spin system on an arbitrary lattice. Spin operators are represented in terms of a two component spinor operator $\binom{a}{b}$ in the Schwinger representation, which is manifestly gauge invariant. Independent boson operators a and b satisfy commutation relations

$$[a, a^\dagger] = [b, b^\dagger] = 1 \tag{2.1}$$

and otherwise vanish. The Hilbert space is spanned by eigenstates of number operators

$$|n_a, n_b\rangle = \frac{1}{\sqrt{n_a!}}\frac{1}{\sqrt{n_b!}}(a^\dagger)^{n_a}(b^\dagger)^{n_b}|0\rangle \tag{2.2}$$

where $|0\rangle$ is defined by $a|0\rangle = b|0\rangle = 0$. The spin operator $\boldsymbol{S}$ is given by

$$\boldsymbol{S} = \tfrac{1}{2}(a^\dagger\, b^\dagger)\boldsymbol{\sigma}\begin{pmatrix} a \\ b \end{pmatrix}$$

where $\boldsymbol{\sigma}$ is the Pauli matrix. A more explicit form is given by (1.3) which satisfies the spin commutation relation in terms of the bosonic commutation relation (2.1). Note that a spin operator is invariant under a phase transformation

$$a \longrightarrow e^{i\alpha}a \qquad b \longrightarrow e^{i\alpha}b \tag{2.3}$$

which is a $U(1)$ gauge transformation. A theory whose Hamiltonian consists of spin operators is gauge invariant. The magnitude of spin is expressed in terms of a number operator N

$$S^2 = \tfrac{1}{2}N\left(\tfrac{1}{2}N + 1\right)$$

where $N = a^\dagger a + b^\dagger b$. Hence, an eigenstate of a maximal 00 commuting set $S^2 = S(S+1)$ and S^z is $|n_a, n_b\rangle$ defined by (2.2) which satisfies

$$S|n_a, n_b\rangle = \tfrac{1}{2}(n_a + n_b)|n_a, n_b\rangle \tag{2.4}$$

$$S^z|n_a, n_b\rangle = \tfrac{1}{2}(n_a - n_b)|n_a, n_b\rangle\,. \tag{2.5}$$

Equation (2.4) indicates that, to obtain a spin s representation, we should take only those states which satisfy a constraint

$$(N - 2s)|s\rangle = 0\,. \tag{2.6}$$

The constraint operator $N - 2s$ generates the $U(1)$ gauge transformation

$$e^{-i\alpha(N-2s)}a\,e^{i\alpha(N-2s)} = e^{i\alpha}a \qquad e^{-i\alpha(N-2s)}b\,e^{i\alpha(N-2s)} = e^{i\alpha}b\,.$$

Now, we discuss elimination of this gauge degree of freedom. In the Schwinger representation, the redundant gauge degree of freedom still exists in the expectation value of a gauge invariant quantity after imposing the constraint (2.6). However, we do not have to take it into account if we employ normalized states, (2.2). This corresponds to dividing the amplitude by the gauge volume in the Fadeev–Popov procedure for a path integral representation. One problem within this framework is how to keep the constraint (2.6) when the model is extended to a lattice model. There are some methods to solve this problem. The Dirac method is well known as one of them [9]. In this method, we eliminate redundant operators from the beginning by introducing a gauge fixing condition. Since one can choose any gauge fixing function if it does not commute with the constraint, one has many choices of gauge fixing. Although different gauge fixing conditions give us different representations, the same result for the expectation value of a gauge invariant quantity should be obtained in any gauge fixing condition. Thus they are equivalent to each other. The same situation can be seen in spin wave theories. Here, we show that both the Holstein–Primakoff and the Dyson–Maleev representations can be regarded as gauge fixed theories with certain gauge fixing conditions following the Dirac method. Both representations can be derived from the Schwinger formalism which is gauge invariant.

Now we describe how to derive these two representations from the Schwinger representation. Let us consider classical coordinates $(a,\ b,\ a^\dagger,\ b^\dagger)$ in the phase space of the Schwinger formalism. To obtain the Holstein–Primakoff representation, we impose a gauge fixing condition

$$a^\dagger = a\,. \tag{2.7}$$

Here, we regard the constraint on the spin magnitude as an equation of classical coordinates

$$a^\dagger a + b^\dagger b - 2s = 0\,. \tag{2.8}$$

We can solve these equations with respect to a and $a^\dagger$ in terms of b and $b^\dagger$

$$a^\dagger = a = \sqrt{2s - b^\dagger b} \tag{2.9}$$

which enables us to eliminate redundant coordinates a and $a^\dagger$. Note that, as well as the constraint, a gauge fixing condition is necessary to restrict the phase space to its sub-space which has even dimensions. Substituting (2.9) into (1.3), we obtain the Holstein–Primakoff representation (1.1). After this elimination procedure one can quantize b and $b^\dagger$ by replacing the Poisson bracket to a commutator. One can check the spin commutation relation in terms of boson operators. At this stage one comes across an ambiguity of operator ordering. According to the usual method to construct an irreducible unitary representation of spin, however, one obtains the unique representation. Although expressions of spin operators in terms of bosons are ambiguous, all representations of the spin are unitary equivalent as far as the spin commutation relation holds. To obtain a complete basis, one defines a highest-weight state by

$$S^+|0\rangle = 0$$

which implies $b|0\rangle = 0$. An eigenstate with an arbitrary eigenvalue can be obtained by applying the operator S^- a certain number of times on the highest-weight state $|0\rangle$. A normalized eigenstate of S^z with an eigenvalue m is

$$|m\rangle = \frac{1}{\sqrt{(s-m)!}}(b^\dagger)^{s-m}|0\rangle \tag{2.10}$$

where $m = s,\ s-1, \ldots, 1-s,\ -s$. This representation possesses a fixed spin magnitude s automatically.

To obtain the Dyson–Maleev representation, we require the following gauge fixing condition

$$a = 1\,. \tag{2.11}$$

This and (2.8) lead to

$$a^\dagger = 2s - b^\dagger b\,. \tag{2.12}$$

The spin operator in the Dyson–Maleev representation (1.2) is obtained by substituting (2.12) into (1.3). The unitary representation of spin is determined uniquely as in the case of the Holstein–Primakoff representation. An eigenstate of S^z with an eigenvalue m and its dual are

$$|m) = \sqrt{\frac{(s+m)!}{(2s)!(s-m)!}}(b^\dagger)^{s-m}|0) \tag{2.13}$$

$$(m| = \sqrt{\frac{(2s)!}{(s+m)!(s-m)!}}(0|b^{s-m} \tag{2.14}$$

where $m = s,\ s-1, \ldots, 1-s,\ -s$. The highest-weight state $|0\rangle$ and its dual $\langle 0|$ are defined by $b|0\rangle = 0$ and $\langle 0|b^\dagger = 0$. These basis are obtained by acting certain number of operators $S^\mp$ to $|0\rangle$ and $\langle 0|$, respectively. $S^+|s)$, $(s|S^-$ and $(-s|S^+$ vanish automatically. The only non-trivial question is whether $S^-|-s)$ vanishes or not. The answer is that it indeed vanishes because $(m|S^-|-s)$ vanishes for any m $(-s \leqslant m \leqslant s)$. Hence, $S^-|-s)$ should be identified with the zero vector. We have changed the definition of an inner product in such a way that $(m|S^+|n)^* = (n|S^-|m)$. On the other hand, $(m|b|n)^* \neq (n|b^\dagger|m)$. Namely, unitarity is preserved in this representation for the spin, but not for the boson. Since there is no representation preserving unitarity for both spin and boson operators in this gauge, we respect the unitarity of gauge invariant operators. This implies that the Dyson–Maleev representation is connected to other gauge fixed theories with a unitary represented boson by an extended $U(1)$ gauge transformation with a complex parameter. The original theory is invariant under this complexified $U(1)$ gauge transformation, which is represented as a non-unitary transformation in the Hilbert space of boson operators. This non-unitary transformation was found already by Oguchi and Honma [6, 7], although the hidden gauge symmetry in spin wave theories has never been discussed until now. Calculations in terms of bosons seems to be complicated for general quantities in this representation. However, a trace of an arbitrary function f of the boson can be calculated in the usual basis defined by (2.10)

$$\mathrm{Tr} f(b, b^\dagger) \equiv \sum_{m=-s}^{s} (m|f(b, b^\dagger)|m) = \sum_{m=-s}^{s} \langle m|f(b, b^\dagger)|m\rangle .$$

Thus, one can calculate thermodynamic quantities in the usual basis even in this representation. $S_{\mathrm{Sch}} S_{\mathrm{HP}}$ and S_{DM} denote spin operators in the Schwinger, the Holstein–Primakoff and the Dyson–Maleev representations, respectively, whose matrix elements agree with each other. Thus, the following equality holds for the trace of an arbitrary function f of spin operator

$$\sum_{n_a+n_b=2s} \langle n_a, n_b| f(\boldsymbol{S}_{\mathrm{Sch}}(a, b, a^\dagger, b^\dagger))|n_a, n_b\rangle$$
$$= \sum_{m=-s}^{s} \langle m| f(\boldsymbol{S}_{\mathrm{HP}}(b, b^\dagger))|m\rangle = \sum_{m=-s}^{s} \langle m| f(\boldsymbol{S}_{\mathrm{DM}}(b, b^\dagger))|m) . \tag{2.15}$$

The equivalence of these representations is obvious.

Now, we extend our method to lattice spin models. To do so, we study the Heisenberg model on an d-dimensional lattice as an example. The Hamiltonian of the Heisenberg model is

$$H = -J \sum_{\langle x, y\rangle} \boldsymbol{S}_x \cdot \boldsymbol{S}_y$$

which is summed over each pair $\langle x, y\rangle$ of nearest neighbour sites. In the Schwinger representation, we introduce two kinds of boson operators a_x and b_x and impose the constraint on states at each lattice site

$$a_x{}^\dagger a_x + b_x{}^\dagger b_x - 2s = 0 .$$

All constraints commute with each other, and each generates the $U(1)$ gauge transformation at the corresponding site. Since the Hamiltonian is invariant under this local gauge transformation, our method can be applied to this model as well. Our method for a single spin model is generalized to that for a lattice spin model straightforwardly. The Holstein–Primakoff or the Dyson–Maleev representations are obtained by certain gauge fixing conditions $a_x^\dagger = a_x$ and $a_x = 1$, respectively. We can eliminate a_x and $a_x^\dagger$ at each lattice site as in the case of a single spin model. The Hamiltonian in the Holstein–Primakoff representation is

$$H_{\mathrm{HP}} = -J \sum_{x,\delta} \left[s\, b_{x+\delta}^\dagger \sqrt{g(n_{x+\delta}) g(n_x)}\, b_x + \tfrac{1}{2} n_{x+\delta}\, n_x - s n_x + \tfrac{1}{2} s^2 \right] \quad (2.16)$$

and the Dyson–Maleev representation is

$$H_{\mathrm{DM}} = -J \sum_{x,\delta} \left[s\, b_{x+\delta}^\dagger\, g(n_x)\, b_x + \tfrac{1}{2} n_{x+\delta}\, n_x - s n_x + \tfrac{1}{2} s^2 \right] \quad (2.17)$$

where

$$g(x) \equiv 1 - \frac{x}{2s} \qquad n_x \equiv b_x^\dagger b_x$$

and δ is summed over each bond connecting a lattice site x to its nearest neighbour sites. Both Hamiltonians, (2.16) and (2.17), are Hermitian, even though (2.17) does not look as if it is. The equality of partition functions in two representations

$$\mathrm{Tr}\, \mathrm{e}^{-\beta H_{\mathrm{HP}}} = \mathrm{Tr}\, \mathrm{e}^{-\beta H_{\mathrm{DM}}} \quad (2.18)$$

is easily understood as in the argument of single spin case.

3. Explicit evaluation in high-temperature expansion

As pointed out in the previous section, the Holstein–Primakoff and the Dyson–Maleev representations are equivalent, because of the gauge invariance of the Schwinger formalism. We have shown that all the matrix elements of spin operators at each lattice site are identical to each other in these representations. However, one can check the equality more explicitly starting with the Hamiltonian given by (2.16) and (2.17), even if one has no knowledge of the gauge invariance. In this section, we show that (2.18) holds exactly to arbitrary orders in high temperature expansion.

To this end, we divide the Hamiltonian H into two parts, a boson number conserving and a non-conserving term:

$$H = H_0 + H_1 .$$

H_0 in both representations is

$$H_0 = J \sum_{x,\delta} \left[-\tfrac{1}{2} n_{x+\delta}\, n_x + s n_x - \tfrac{1}{2} s^2 \right] \quad (3.1)$$

which conserves the number of bosons at each lattice site. On the other hand, the boson number non-conserving part is

$$H_1 = \sum_{x,\delta} v_{x+\delta,x} \quad (3.2)$$

where $\nu_{y,x}$ in the Holstein–Primakoff representation is

$$\nu_{y,x} = -Js\, b_y^\dagger \sqrt{g(n_y)g(n_x)}\, b_x \tag{3.3}$$

while in the Dyson–Maleev representation is

$$\nu_{y,x} = -Js\; b_y^\dagger\, g(n_x)\, b_x\,. \tag{3.4}$$

The partition function Z can be expressed as the trace of a density operator

$$Z = \mathrm{Tr}\, \mathrm{e}^{-\beta(H_0+H_1)}\,.$$

High temperature expansion can be done around $\beta = 0$ as follows:

$$Z = \sum_{n=0}^{\infty} \frac{(-\beta)^n}{n!} \mathrm{Tr}(H_0 + H_1)^n\,. \tag{3.5}$$

Let us expand the term in the nth order

$$\mathrm{Tr}(H_0 + H_1)^n = \sum_{(l_1,\ldots,l_k,m_1,\ldots,m_k)} \mathrm{Tr}[H_0{}^{l_1} H_1{}^{m_1} H_0{}^{l_2} H_1{}^{m_2} \ldots H_0{}^{l_k} H_1{}^{m_k}]\,. \tag{3.6}$$

The sum is over each natural number partition $(l_1, \ldots, l_k, m_1, \ldots, m_k)$ of n which obeys $l_1 + \cdots + l_k + m_1 + \cdots + m_k = n$ and $l_i, m_i \geqslant 0$. Only those terms having the same number of b_x and $b_x^\dagger$ at each lattice site can survive in (3.6), since the number of bosons is conserved at each lattice site in the trace. Thus, one term with a partition of n can be represented in bosonic random walk with m steps ($m = m_1 + \cdots + m_k$):

$$\mathrm{Tr}\Big[H_0{}^{l_1} H_1{}^{m_1} H_0{}^{l_2} H_1{}^{m_2} \ldots H_0{}^{l_k} H_1{}^{m_k}\Big] = (-Js)^m \sum_{\{x_1,\ldots,x_m,y_1,\ldots,y_m\}} \mathrm{Tr}\Big[H_0{}^{l_1} \nu_{y_1,x_1} \cdots \\ \cdots \nu_{y_{m_1},x_{m_1}} H_0{}^{l_2} \nu_{y_{m_1+1},x_{m_1+1}} \cdots H_0{}^{l_k} \ldots \nu_{y_m,x_m}\Big] \\ . \tag{3.7}$$

The sum is over set of nearest neighbour sites pairs $\{\langle x_1, y_1\rangle, \ldots, \langle x_{m_1}, y_{m_1}\rangle, \ldots, \langle x_m, y_m\rangle\}$ which becomes a closed path on the d-dimensional lattice.

Now, we show an equality of partition functions calculated in the Holstein–Primakoff and the Dyson–Maleev representations. It is sufficient if we show the equality for one arbitrary term in the right-hand side of (3.7) with one fixed set of nearest neighbour sites pairs. This can be done by rewriting the order of operators from the one representation to the other. First, we rewrite parts with respect to only one lattice site x. Using relations for an arbitrary function f

$$f(n_x - 1)b_x^\dagger = b_x^\dagger f(n_x) \qquad f(n_x + 1)b_x = b_x f(n_x) \tag{3.8}$$

the following equation can be shown

$$\mathrm{Tr}\Big[\ldots \Big(\sqrt{g(n_x)}b_x\Big)^{p_1} \ldots \Big(b_x^\dagger \sqrt{g(n_x)}\Big)^{q_1} \ldots \Big(\sqrt{g(n_x)}b_x\Big)^{p_i} \ldots \Big(b_x^\dagger \sqrt{g(n_x)}\Big)^{q_i} \ldots\Big] \\ = \mathrm{Tr}\Big[\ldots \big(g(n_x)b_x\big)^{p_1} \ldots (b_x^\dagger)^{q_1} \ldots \big(g(n_x)b_x\big)^{p_i} \ldots (b_x^\dagger)^{q_i} \ldots\Big] \tag{3.9}$$

if $\sum_{j=1}^{i} p_j = \sum_{j=1}^{i} q_j$. We have neglected terms which commute with n_x. Equation (3.9) holds because $\sqrt{g(n_x)}$ can jump from one place behind $b_x^\dagger$ to the other in front of b_x without any change if $\sqrt{g(n_x)}$ jumps over the same number of $b_x^\dagger$ and b_x. This argument can apply to each lattice site again and again, and finally the equality is proved for one arbitrary term in the right-hand side of (3.7). And then, the equality between two representation is verified for (3.6). Thus, the partition functions in the Holstein–Primakoff and the Dyson–Maleev are identical with each other to arbitrary orders in high-temperature expansion.

4. Summary

We have shown that several types of spin wave theories, the Holstein–Primakoff, the Dyson–Maleev and the Schwinger representations are equivalent to each other. We point out that the Schwinger formalism possesses a local $U(1)$ gauge invariance. The Holstein–Primakoff and the Dyson–Maleev are obtained from the Schwinger formalism by different gauge fixing conditions, following the Dirac method for a gauge invariant theory. Matrix elements of gauge invariant operators, such as a spin operator, agree with each other in these three representations. In the Dyson–Maleev representation however, the boson operator $b^\dagger$ is not Hermitian conjugate of b unlike the Holstein–Primakoff representation. This fact implies that the Dyson–Maleev representation is connected with the Holstein–Primakoff representation by a complexified $U(1)$ gauge transformation. We can show the equality of partition functions between the Holstein–Primakoff and the Dyson–Maleev representations more explicitly in the Heisenberg model as an example. High-temperature expansion enables us to show the equality of partition function in two representations to arbitrary orders. We conclude that three spin wave theories the Schwinger, the Holstein–Primakoff and Dyson–Maleev are equivalent to each other.

Acknowledgments

We would like to thank H Mukaida and N Sakai for enlightening discussions at the early stage of this work. We are grateful to T Fujita for reading the manuscripts.

References

[1] Holstein T and Primakoff H 1940 *Phys. Rev.* **58** 1098
[2] Dyson F J 1956 *Phys. Rev.* **102** 1217, 1230
[3] Maleev S V 1958 *JETP* **6** 776
[4] Schwinger J 1952 *On Angular Momentum*; later published in 1965 *Quantum Theory of Angular Momentum* ed L C Biedenharn and H Van Dam (New York: Academic) p 229
[5] Takahashi M 1989 *Phys. Rev.* B **40** 2494
[6] Oguchi T 1961 *Prog. Theor. Phys.* **25** 721
[7] Oguchi T and Honma A 1963 *J. App. Phys.* **34** 1153
[8] Arovas D P and Auerbach A 1988 *Phys. Rev.* B **38** 316; 1988 *Phys. Rev. Lett.* **61** 617
[9] Dirac P A M 1964 *Lectures on Quantum Mechanics* Belder Graduate School of Science, Yeshiva University